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THE SIMPLIFIED REDUCTION FORMULA TO EVALUATE THE INTEGRAND OF $x^n e^x$

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Abstract. Certain integrals involving powers of the variable or powers of functions of the variable can be related to integrals of the same form but containing reduced powers and such relations are called REDUCTION FORMULAS. Successive use of such formulas will often allow a given integral to be expressed in terms of a much simpler one. In this paper we present the simplified reduction formula for evaluation integrals of the form $\int x^n e^x dx$.

1.0 Introduction

Previously the technique of integration by parts as a means of integrating products has been explored. The technique is reliant on reducing the original integral to the difference of a product and another integral of simpler form. Often the reduced integral is of the same form as the original integral. Reduction formulae can be written for many common integrals and can simplify the process of integration.

2.0 Method

Consider the integral $\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx$ i.e. if $I_n = \int x^n e^x dx$, then $I_n = x^n e^x - nI_{n-1}$. This can be expressed as:

$$\int x^n e^x dx = x^n e^x - nx^{n-1} e^x + n(n-1)x_{n-2} e^x - n(n-1)(n-2)x^{n-3} e^x + \dots\dots\dots + (-1)^n e^x + C \qquad (1)$$

Where C is a constant of integration. Equation (1) can be written as

$$\int x^n e^x dx = n!\left[\frac{x^n}{n!} - \frac{x^{n-1}}{(n-1)!} + \frac{x^{n-2}}{(n-2)!} - \frac{x^{n-3}}{(n-3)!} + \frac{x^{n-4}}{(n-4)!} \dots\dots\dots + \frac{(-1)^n}{(n-n)!}\right] e^x + +C \qquad (2)$$

Similarly, if $n \in Z^+$ then equation (2) can be written as

$$\int x^n e^x dx = n! e^x \sum_{i=0}^{n} (-1)^i \frac{x^{n-i}}{(n-1)!} + C \qquad (3)$$

Equation (3) above gives the **SIMPLIFIED REDUCTION FORMULA**. Once the learners grasp it, learners can easily write down the integrand with a lot of ease as no much algorithm is needed (i.e. multiplication is minimized).

NUMERICAL EXAMPLES

2.1 Example 1

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Use the simplified reduction formula to evaluate the following:

i. $\int x^4 e^x dx$

ii. $\int x^7 e^x dx$

Solution

i. Here $n = 4 \in Z^+$

$$\int x^4 e^x dx = 4! e^x \sum_{i=0}^{4} (-1)^i \frac{x^{n-i}}{(n-i)!} + C = 24 e^x \left[\frac{x^4}{24} - \frac{x^3}{6} + \frac{x^2}{2} - x + 1 \right] + C$$

ii. Here $n = 7 \in Z^+$

$$\int x^7 e^x dx = 7! e^x \left[\sum_{i=0}^{7} (-1)^i \frac{x^{n-i}}{(n-i)!} \right] + C = 5040 \; e^x \left[\frac{x^7}{5040} - \frac{x^6}{720} + \frac{x^5}{120} - \frac{x^4}{24} + \frac{x^3}{6} - \frac{x^2}{2} + x - 1 \right] + C$$

2.2 REMARKS

1. In our expansion if $n \in Z$ and n is an even number; then terms associated with even factorials are given precedence, i.e. assigned positive sign whereas those with odd factorials assigned negative sign.
2. If $n \in Z$ and n is an odd number; then terms associated with odd factorials are given precedence, i.e. assigned positive sign whereas those with even factorials assigned negative sign.
3. The already obtained simplified reduction formula can be extended to evaluate integrand of the form $\int x^n e^{mx} dx$ as follows; (where $m \in \Re$ and $m \neq 0, n \in Z^+$).

$$\int x^n e^{mx} dx = \frac{x^n e^{mx}}{m} - \frac{n x^{n-1} e^{mx}}{m^2} + \frac{n(n-1) x^{n-2} e^{mx}}{m^3} - \frac{n(n-1)(n-2) x^{n-3} e^{mx}}{m^4} + \ldots\ldots \tag{4}$$

Or

$$\int x^n e^{mx} dx = \frac{n!}{m} \left[\frac{x^n}{n!} - \frac{x^{n-1}}{(n-1)!\, m} + \frac{x^{n-2}}{(n-2)!\, m^2} - \frac{x^{n-3}}{(n-3)!\, m^3} + \ldots. + \frac{(-1)^n}{(n-n)!\, m^n} \right] e^{mx} + C$$

$$= \left[\frac{n!}{m} e^{mx} \sum_{i=0}^{n} \frac{(-1)^i x^{n-i}}{(n-i)!\, m^i} \right] + C$$

$$= \left[\frac{n!}{m} \sum_{i=0}^{n} \frac{(-1)^i x^{n-i}}{(n-i)!\, m^i} \right] e^{mx} + C \tag{5}$$

Equation (5) gives the ***Extended Simplified Reduction Formula***.

2.3 Example 2

Use the Extended Simplified Reduction Formula to evaluate

$\int x^5 e^{2x} dx$

Solution

Here $n = 5 \; m = 2$

$$\int x^5 e^{2x} dx = \left[\frac{5!}{2} \sum_{i=0}^{5} \frac{(-1)^i x^{5-i}}{(5-i)! m^i} \right] e^{2x} + C$$

$$= \frac{120}{2} \left[\frac{x^5}{120} - \frac{x^4}{(24)2} + \frac{x^3}{6(2)^2} - \frac{x^2}{2(2)^3} + \frac{x}{(2)^4} - \frac{1}{(2)^5} \right] e^{2x} + C$$

$$= 60 \left[\frac{x^5}{120} - \frac{x^4}{48} + \frac{x^3}{24} - \frac{x^2}{16} + \frac{x}{16} - \frac{1}{32} \right] e^{2x} + C$$

2.4 Example 3

Find $\int x^6 e^{-3x} dx$

Solution

Here $n = 6 \; m = -3$

$$\int x^6 e^{-3x} dx = \left[\frac{6!}{-3} \sum_{i=0}^{6} \frac{(-1)^i x^{6-i}}{(6-i)! m^i} \right] e^{-3x} + C$$

Or

$$\int x^6 e^{-3x} dx = \frac{720}{-3} \left[\frac{x^6}{6!} - \frac{x^5}{5!(-3)^1} + \frac{x^4}{4!(-3)^2} - \frac{x^3}{3!(-3)^3} + \frac{x^2}{2!(-3)^4} - \frac{x}{(-3)^5} \frac{1}{(-3)^6} \right] e^{-3x} + C$$

$$= -240 \left[\frac{x^6}{720} - \frac{x^5}{-360} + \frac{x^4}{216} - \frac{x^3}{-81} + \frac{x^2}{162} - \frac{x}{-243} + \frac{1}{729} \right] e^{-3x} + C$$

$$= -240 \left[\frac{x^6}{720} + \frac{x^5}{360} + \frac{x^4}{216} + \frac{x^3}{81} + \frac{x^2}{162} + \frac{x}{243} + \frac{1}{729} \right] e^{-3x} + C$$

3.0 Points to note

i. If $m > 0$ (*positive*) then the terms have sign corresponding to remarks (i) and (ii)

ii. If $m < 0$ then all the terms will have negative sign; this can act as a confirmation test on the accuracy of the integrand.

4.0 Conclusion

The Simplified Reduction Formula and Extended Reduction Formula are great significance since they simplify the computation process involved in expanding the integrand. This is an important aspect to any mathematical formula.

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